Gauss map

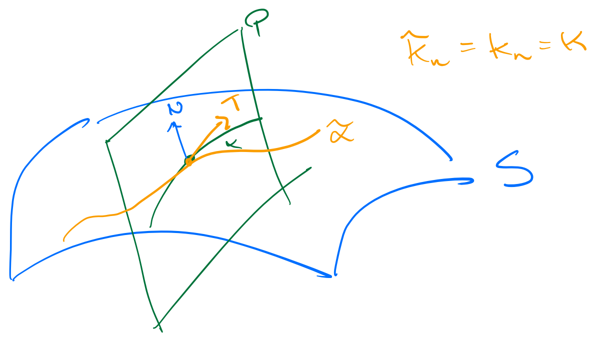
Prologue: how can we use curves to understand surfaces?

A Normal slicing

Recall k_n is the normal curvature of α in S, and is given by

$$k_n = \langle T', N \rangle$$

Observation: if P is a plane through $q \in S$ containing N_q, and $\alpha = P \cap S$, then at q, T' is parallel to N, so $k_n = \kappa$. $(k_g = 0)$



By the following prop, this is enough to describe the normal curvature of any curve through q.

Prop (Meusnier) All curves α in S with the same tangent line have the same normal curvatures.

A priori, k_n depends on the first 2 derivatives of α b/c

$$k_n = \langle T', N \rangle = \langle \alpha'', N \rangle$$

T $k_n = -\langle T, N' \rangle$

$N' = \frac{dN}{dt}$

$= dN_p(\alpha') \cdot \frac{d\alpha}{dt}$

Recall (Darboux frame) N depends only on S, not on α, i.e. $N(t)$ factors

$$I \xrightarrow{\alpha} S \xrightarrow{N} S^2 \subseteq \mathbb{R}^3$$

$k_n = -\langle T, dN(T)\rangle$ only depends on T $\lrcorner$

- You can think of it as a partial framing of S.
- If you want to be careful about domains + codomains,

$$T \in T_pS, \quad dN_p : T_pS \to T_{N(p)}S^2 \quad \text{canon isom.}$$

(really, N is the Maurer-Cartan form of the Lie group $\mathbb{R}^3$

$$p^{-1}dp(N) : T_p\mathbb{R}^3 \to T_0\mathbb{R}^3.$$

dN was M-C form again?.)

Note that this says the normal curvatures behaves like a quadratic form

Dfn The quadratic form II_p defined on T_pS by $II_p(v) = -\langle dN_p(v), v\rangle$ is called the second fundamental form of S at p.

Rmk here $\langle \, , \rangle$ is the induced Riemannian metric, and $I_p(v) = \langle v, v\rangle$ is the first fundamental form.

More linear algebra: $V = T_pS$, $g = \langle \, , \rangle$ = Riem metric.

Three equivalent lin-alg. objects

1. Quadratic forms Q on V
2. Symmetric bilinear forms on V
3. self-adjoint endomorphisms of V $g(Bv, w) = g(v, Bw)$ $\forall v, w$.

(1) $\leftrightarrow$ (2)

quadratic forms $\longleftrightarrow$ symmetric bilinear forms

$$Q_h(v) = h(v,v) \longleftarrow h \in \mathrm{Sym}^2 V^*$$

$$Q \longmapsto h_Q(v,w) = \frac{Q(v+w) - Q(v) - Q(w)}{2}$$

(2) $\leftrightarrow$ (3)

$g = g_{ij} \in \mathrm{Sym}^2 V^* \subseteq V^* \otimes V^*$

$g^{-1} := g^{ij} \in \mathrm{Sym}^2 V \subseteq V \otimes V$ defined by $g g^{-1} = \delta$

i.e. the inverse matrix

$$h(v,w) = g(Bv, w)$$

$$h_{ij} \longmapsto B^k{}_j = g^{ki} h_{ij}$$

$$h_{ij} = g_{ik} B^k{}_j \longleftarrow B^k{}_j$$

<u>Thm</u> (Spectral theorem) If B is self-adjoint, it has an orthonormal basis of eigenvectors.

This all fits our situation very nicely once we know:

<u>Prop</u> dN_p is self-adjoint.

$\ulcorner$ (cf. Meusnier's thm.)

Pick a chart φ for S at p

$$-\langle dN\circ\varphi_1, \varphi_2\rangle = -\left\langle \frac{\partial(N\circ\varphi)}{\partial x^1}, \frac{\partial\varphi}{\partial x_2}\right\rangle$$
$$= \left\langle N\circ\varphi, \frac{\partial^2\varphi}{\partial x_1\partial x_2}\right\rangle \quad \lrcorner$$

<u>Cor</u>

there is an orthonormal basis $\{e_1, e_2\}$ of T_pS such that

- $dN_p(e_i) = -k_i e_i \qquad (i=1,2)$
- $k_1 = \min\limits_{I_p(v)=1} II_p(v)$, $k_2 = \max\limits_{I_p(v)=1} II_p(v)$

<u>Dfns</u> Shape operator, principal curvatures.

In analogy with Frenet frames, information about how the surface bends in space is contained in dN.

<u>Dfn</u>
- k_1 and k_2 are called the <u>principal curvatures</u> of S at p.
- If $k_1=k_2$, p is called an <u>umbilic point</u> of S
- If p is not umbilic, e_1 and e_2 are unique up to sign and $\pm e_1$ and $\pm e_2$ are called the <u>principal</u>

Definitions.

- $\frac{k_1+k_2}{2}$ is called the mean curvature of S at p, and is written H. $\quad dN = \begin{bmatrix} -k_1 & 0 \\ 0 & -k_2 \end{bmatrix}$ in the basis $\{e_1, e_2\}$

  Note $H = -\frac{1}{2}\operatorname{tr}(dN)$.

- $k_1 k_2$ is called the Gauss curvature of S at p and is written K.

  Note $K = \det(dN)$.

Eg's

- N of S^2 is $\pm$ id
- N of $S^1\times\mathbb{R}$ is the map to the equator

Extended ex: saddle $\quad z = \frac{1}{2}(y^2 - x^2)$

Parametrize $\quad \varphi(x_1, x_2) = (x_1, x_2, \frac{1}{2}(x_2^2 - x_1^2))$

$$\varphi_1 = (1, 0, -x_1)$$
$$\varphi_2 = (0, 1, x_2)$$

Recall $\quad (N\circ\varphi) = \frac{\varphi_1\times\varphi_2}{|\varphi_1\times\varphi_2|} = \frac{(x_1, -x_2, 1)}{\sqrt{1+x_1^2+x_2^2}}$

skip

$$(N\circ\varphi)_1 = \left(\frac{1}{\sqrt{}} - \frac{x_1^2}{\sqrt{}^3}, \frac{x_2x_1}{\sqrt{}^3}, \frac{-x_1}{\sqrt{}^3}\right)$$

$$= (1+x_2^2, x_2x_1, -x_1)/\sqrt{}^3$$

$$= \frac{1+x_2^2}{\sqrt{}^3}\varphi_1 + \frac{x_1x_2}{\sqrt{}^3}\varphi_2$$

(Does 3rd coord work?)

$$(N\circ\varphi)_2 = \frac{-x_1x_2}{\sqrt{}^3}\varphi_1 - \frac{(1+x_1^2)}{\sqrt{}^3}\varphi_2$$

$$B = \frac{1}{\sqrt{\ }^{3}} \begin{bmatrix} 1+x_2^2 & x_1 x_2 \\ -x_1 x_2 & -1-x_1^2 \end{bmatrix}$$

$$\Rightarrow H = \frac{x_2^2 - x_1^2}{\sqrt{\ }^{3}}$$

$$K = \frac{-1}{\sqrt{\ }}$$

<u>Alt</u> $K = \dfrac{\det(h_{ij})}{\det(g_{ij})}$

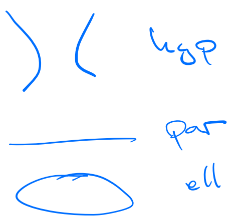
Friday Jul 28 9-11 1.5 hours

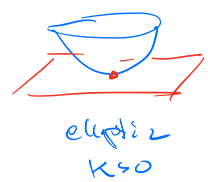
INF 230 gHS

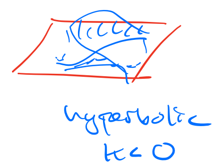
Let k_1, k_2 be the principle curvatures of S at a point p. p is called

	if	i.e.
hyperbolic	$k_1<0,\ k_2>0$	$K<0$
parabolic	exactly one is 0	$K=0,\ H\neq 0$
elliptic	k_1 and k_2 have the same sign	$K>0$
planar	$k_1=k_2=0$	$K=H=0$

$\{\mathrm{II}(v)=1\}$







Prop1 If $p\in S$ is elliptic, then $\exists$ nbhd V on which all pts of S lie on one side of T_pS as a affine plane

If $p\in S$ is hyperbolic, then $\forall\ V$, $\exists$ points in V on each side of T_pS

Pf Use param X w/ $X(0,0)=p$

$d(u,v)=\langle X(u,v)-X(0,0), N(p)\rangle$ is dist to plane

Taylor expand X

$$X(u,v)=X(0,0)+\underset{X_u(0,0)}{\underbrace{X_u}}u+X_v v+\tfrac{1}{2}\left(X_{uu}u^2+2X_{uv}uv+X_{vv}v^2\right)+R$$

where $R=o(u^2+v^2)$

$$\Rightarrow d(u,v)=\tfrac{1}{2}\left(\langle X_{uu},N\rangle u^2+2\langle X_{uv},N\rangle uv+\langle X_{vv},N\rangle v^2\right)+\langle R,N\rangle$$

$$=\tfrac{1}{2}\mathrm{II}\left(uX_u^{(0)}+vX_v^{(0)}\right)+\langle R,N\rangle$$

(Recall

$\mathrm{II}(X_u)=-\langle X_u, dN(X_u)\rangle$

$=-\langle X_u, N_u\rangle$

$=\langle X_{uu}, N\rangle$

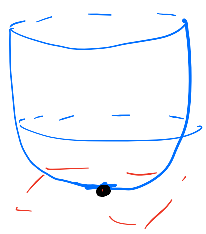
etc.

b/c $R = o(u^2+v^2)$, $\dfrac{d(u,v)}{u^2+v^2} \longrightarrow \dfrac{\frac{1}{2}\mathrm{II}(uX_u+vX_v)}{u^2+v^2}$ as $u^2+v^2 \to 0$

so, if p is elliptic, with $k_1>0$ WLOG, the limit is $\geq k_1 > 0$ in all directions, so $d>0$ for u,v small.

and if p is hyperbolic, the limit is positive in the $\pm e_2$ direction and negative in the $\pm e_1$ direction, so the same is true for d for u,v small. $\lrcorner$

Rmk: Converse is false, eg surface of revolution of $z=y^4$



$\leftarrow$ $dN = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
(planar point)

Eg: Torus (HW)